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Report Number:

94-057

Hadjidimos, A.; Noutsos, D.; and Tzoumas, M., "Exact SOR Convergence Regions for a General Class of p-Cyclic Matrices" (1994). *Department of Computer Science Technical Reports*. Paper 1157.
<https://docs.lib.purdue.edu/cstech/1157>

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**EXACT SOR CONVERGENCE REGIONS FOR
A GENERAL CLASS OF p -CYCLIC MATRICES**

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**CSD-TR-94-057
August 1994**

Exact SOR Convergence Regions for a General Class of p-Cyclic Matrices

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Abstract

Linear systems whose associated block Jacobi iteration matrix B is weakly cyclic generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_p)$ in the spirit of Li and Varga are considered. Regions of convergence for the corresponding block p -cyclic SOR method are derived and the exact convergence domains for real spectra, $\sigma(B^p)$, of the same sign are obtained. Moreover, analytical expressions for two special cases for $p = 5$ are given. The tools used for this work are mainly from complex analysis and extensive use of (asteroidal) hypocycloids in the complex plane is made to produce our results.

Key Words: p-cyclic matrices, SOR method, hypocycloids.

Running Title: Convergence Regions for p-Cyclic SOR

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1. Introduction and Preliminaries

Suppose that $A \in \mathcal{U}^{n,n}$ is partitioned in a $p \times p$ block form and its diagonal blocks A_{ii} , $i = 1(1)p$, are square and nonsingular. Suppose also that the associated block Jacobi iteration matrix

$$B := I - D^{-1}A, \quad (1.1)$$

with $D := \text{diag}(A_{11}, A_{22}, \dots, A_{pp})$, is weakly cyclic generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_p)$ in the spirit of Li and Varga [12]. According to their definition: “*The $p \times p$ block matrix B is a weakly cyclic matrix generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_p)$ of the integers $\{1, 2, \dots, p\}$ iff*

$$B_{\sigma_j \sigma_{j+1}} \neq 0, \quad j = 1(1)p, \quad \sigma_{p+1} = \sigma_1 \quad (1.2)$$

and

$$B_{ij} \equiv 0, \quad \text{otherwise.} \quad (1.3)$$

We note here that the block Jacobi matrices associated with a class of block p cyclic consistently ordered matrices [21], [25], [1] are generated, according to the above definition from $\sigma = (p, p-1, p-2, \dots, 1)$ while those of a class of the $(q, p-q)$ -generalized consistently ordered matrices (or $(q, p-q)$ -GCO) (see [25]) are generated from $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_p)$ with $1 \leq \sigma_j \leq p$, $j = 1(1)p$, and $\sigma_{j+1} = p - q + \sigma_j$ or $\sigma_{j+1} = \sigma_j - q$. So, the definition presented above is the most general one for the family of block p -cyclic matrices which have only p nonidentically zero blocks and diagonal blocks square nullmatrices. Obviously, the block graph of the block matrix B is a cycle.

For the solution of the nonsingular linear system

$$Ax = b \quad (1.4)$$

the Successive Overrelaxation (SOR) iterative method

$$x^{(m+1)} = \mathcal{L}_\omega x^{(m)} + c_\omega, \quad m = 0, 1, 2, \dots \quad (1.5)$$

is considered, where $x^{(0)} \in \mathcal{U}^n$ is arbitrary, ω is the relaxation factor and

$$\begin{aligned} \mathcal{L}_\omega &:= (I - \omega L)^{-1}[(1 - \omega)I + \omega U], \\ c_\omega &:= \omega(I - \omega L)^{-1}D^{-1}b \end{aligned} \quad (1.6)$$

with L and U being the strictly lower and the strictly upper triangular components of the Jacobi matrix B , respectively.

In case A belongs to the family of matrices mentioned previously the sets of the eigenvalues $\mu \in \sigma(B)$ and $\lambda \in \sigma(\mathcal{L}_\omega)$ are connected via the functional equation

$$(\lambda + \omega - 1)^p = \omega^p \mu^p \lambda^{|\zeta_L|} \quad (1.7)$$

Equation (1.7) is a special case of the functional equation

$$[\lambda - (1 - \omega)(1 - \hat{\omega})]^p = \lambda^k [\lambda\omega + \hat{\omega} - \omega\hat{\omega}]^{|\zeta_L| - k} \times [\lambda\hat{\omega} + \omega - \omega\hat{\omega}]^{|\zeta_U| - k} (\omega + \hat{\omega} - \omega\hat{\omega})^{2k} \mu^p \quad (1.8)$$

which connects the eigenvalues μ of B and λ of the Unsymmetric Successive Overrelaxation (US-SOR) iteration operator $T_{\omega, \hat{\omega}}$. Equation (1.8) is due to Li and Varga [12]. In (1.8), $|\zeta_L|$, $|\zeta_U|$ and k are integers which are well-defined in [12] (see also [18]). More specifically, from the definitions in [12], $|\zeta_L|$ and $|\zeta_U|$ are the numbers of the nonzero blocks in the block triangular matrices L and U while k is the number of nonzero blocks of the matrix product LU . Note that (1.7) is derived from (1.8) for $\hat{\omega} = 0$, with $\hat{\omega}$ being the underrelaxation parameter.

The functional equation (1.7) generalizes the one for $(q, p - q)$ -GCO matrices of Verner and Bernal [22], which was first mentioned by Varga (see [21]). It is derived from (1.7) for $|\zeta_L| = p - q$. In this work we study equation (1.7), where, without loss of generality, we assume that $|\zeta_L| = p - q$ and that $\text{g.c.d.}(p, q) = 1$.

For the study of the convergence properties of the SOR method, in the case we are interested in, information about the spectrum of B , $\sigma(B)$, may enable us to answer one or more of the following questions:

- (i) What is the largest region, in the complex plane, containing $\sigma(B)$ for which (1.4) converges?
- (ii) For what pairs $(\rho(B), \omega)$ does (1.4) converge? and
- (iii) What is the (optimal) value of ω that minimizes $\rho(\mathcal{L}_\omega)$ for a given $\rho(B)$?

Complete answers to the three questions above have only been given for very particular classes of p -cyclic matrices. Answers to question (i) have been given for the class of block p -cyclic consistently ordered matrices, among others, by Young [24], (see also [25]), Varga [20], (see also [21]), Niethammer and Varga [15], Galanis, Hadjidimos, and Noutsos [3], [4], [5], Wild and Niethammer

[23], Eiermann, Niethammer, and Ruttan [2], Kredell [11], Niethammer [14], Kontovassilis, Plemmons, and Stewart [10], Noutsos [17], and Hadjidimos and Plemmons [9]. Answers to question (ii) have again been given for the block p -cyclic consistently ordered matrices by many researchers (see, e.g., [24], [7], for nonnegative spectra $\sigma(B^p)$, as well as [11], [14], [16], [3], [4], [5], [23], for nonpositive spectra $\sigma(B^p)$ and [7], [23], [17] for both nonnegative and nonpositive spectra). For block $(q, p-q)$ -GCO matrices $q \neq p-1$ by Nichols and Fox [13] for nonnegative spectra, by Galanis, Hadjidimos, Noutsos and Tzoumas [6] for nonpositive spectra in the case of $(p-1, 1)$ -GCO matrices and by Hadjidimos, Noutsos and Tzoumas [8] for both cases. Finally, answers to question (iii) for the block p -cyclic consistently ordered matrices have been given in [24], [11], [14], for $p = 2$, in [16] for $p = 3$, and in [20], [5], [23], and other works, for any p .

In the present work we shall try to give an answer to the first two questions raised for the general case of the family of matrices A considered in the beginning. We organize our work as follows. In Section 2 the answer to question (i) is given, where among the well-known hypocycloidal curves used in the analysis asteroidal ones are also considered. In Section 3 we study the domains of converge in the $(\rho(B), \omega)$ -plane. Finally, in Section 4 we study in detail the particular cases $(p, q) = (5, 2)$ and $(5, 3)$. As will be seen some new results are obtained and some known ones are recovered.

2. Hypocycloids and Regions of convergence

We begin our analysis with the functional eigenvalue relationship (1.7) by replacing $|\zeta_L|$, with $p - q$, and we consider that $\omega \in (1, 2)$. The analysis for the case $\omega \in (0, 1)$ is similar. Using the transformation

$$\phi = \frac{1}{\lambda} \quad (2.1)$$

and substituting in (1.7) we obtain

$$(1 - (1 - \omega)\phi)^p = \omega^p \mu^p \phi^q. \quad (2.2)$$

Our objective is to find the smallest region in the complex plane containing the eigenvalues $\mu \in \sigma(B)$ and which has an image, through the mapping (2.2), in the exterior of the circle ∂D_η (where $D_\eta := \{\phi : \phi = \eta e^{i\theta}, \eta > 0, \theta \in [0, 2\pi)\}$), or, equivalently, in the interior of $\partial D_{1/\eta}$ since $\lambda = \frac{1}{\eta} e^{-i\theta}$. Then the spectral radius of the SOR iteration matrix, $\rho(\mathcal{L}_\omega)$, will be less than or equal to $\frac{1}{\eta}$, with equality holding iff there is an eigenvalue of B on the boundary of the region to be found.

Extracting p^{th} roots of both sides of (2.2) we obtain

$$1 - (1 - \omega)\phi = \omega\mu(\phi^q)^{1/p} = \omega\mu\phi^{q/p}. \quad (2.3)$$

and the corresponding region for μ is defined by the transformation

$$z := \frac{1 - (1 - \omega)\phi}{\omega\phi^{q/p}}. \quad (2.4)$$

To study the mapping in (2.4) we use the transformation

$$\zeta := \phi^{q/p} \quad (2.5)$$

and then (2.4) becomes

$$z := z(\zeta) = \frac{1 - (1 - \omega)\zeta^{p/q}}{\omega\zeta}. \quad (2.6)$$

From (2.5) we have that the closed disk \overline{D}_η , $|\phi| \leq \eta$, is mapped onto the sectors

$$S_k = \{\rho^{q/p} e^{i(2k\pi + \theta)q/p} : \rho \in [0, \eta], \theta \in [0, 2\pi)\}, \quad k = 0(1)p - 1$$

or, equivalently, onto

$$S_k = \{\rho^{q/p} e^{i(\frac{2kq\pi}{p} + \theta)} : \rho \in [0, \eta], \theta \in [0, \frac{2q\pi}{p})\}, \quad k = 0(1)p - 1. \quad (2.7)$$

Consequently, we have to study the transformation (2.6) for each one of the above sectors. However, since from (2.6), if $\zeta' = \zeta e^{i\frac{2kq\pi}{p}}$, $k = 1(1)p - 1$, then for the images z' of ζ' and z of ζ there holds $z' = z e^{-i\frac{2iq\pi}{p}}$, it suffices to study (2.6) for S_0 only. Let then \overline{S}_0 be the associated closed sector, i.e.,

$$\overline{S}_0 = \{\rho^{q/p} e^{i\theta} : \rho \in [0, \eta], \theta \in [0, \frac{2q\pi}{p}]\}. \quad (2.8)$$

Obviously, the boundary of \overline{S}_0 is the closed curve

$$\partial S_0 = \{\rho : \rho \in [0, \eta]\} \cup \{\rho e^{i\frac{2q\pi}{p}} : \rho \in [0, \eta]\} \cup \{\eta e^{i\theta} : \theta \in [0, \frac{2q\pi}{p}]\} \quad (2.9)$$

The image of ∂S_0 by means of (2.6) is a closed curve that consists of the union of the images of the three segments in (2.9). Since the second line segment in (2.9) is obtained from the first one

by a rotation about the origin through an angle $\frac{2q\pi}{p}$ it suffices to study the first line segment and the third curved one. For this we have:

Lemma 2.1. The image of the line segment $\{\rho : \rho \in [0, \eta]\}$ via (2.6) is an infinite line segment on the positive real semiaxis. Moreover, if $\eta \leq \hat{\eta} := \left[\frac{q}{(p-q)(\omega-1)} \right]^{q/p}$ the mapping in question is a $1-1$ onto the line segment $\left[\frac{1-(1-\omega)\eta^{p/q}}{\omega\eta}, \infty \right]$. On the other hand if $\eta \geq \hat{\eta}$ the image is the line segment $\left[\frac{1-(1-\omega)\hat{\eta}^{p/q}}{\omega\hat{\eta}}, \infty \right]$ and the corresponding mapping is not a $1-1$ one.

Proof: Differentiating (2.6), for $\zeta = \rho$, with respect to (wrt) ρ , it is readily checked that $\frac{\partial z}{\partial \rho} < 0$ for $\rho < \hat{\eta}$, $\frac{\partial z}{\partial \rho} = 0$ at $\rho = \hat{\eta}$, and $\frac{\partial z}{\partial \rho} > 0$ for $\rho > \hat{\eta}$. Therefore, for ρ increasing continuously from 0 to $\eta \leq \hat{\eta}$ the corresponding images decrease continuously from ∞ to $\frac{1-(1-\omega)\eta^{p/q}}{\omega\eta}$ and the mapping is $1-1$. On the other hand, for $\eta > \hat{\eta}$ the images of ρ , increasing from 0 to $\hat{\eta}$, decrease from ∞ to $\frac{1-(1-\omega)\hat{\eta}^{p/q}}{\omega\hat{\eta}}$ and as ρ keeps on increasing up to η , its images increase from $\frac{1-(1-\omega)\hat{\eta}^{p/q}}{\omega\hat{\eta}}$ to $\frac{1-(1-\omega)\eta^{p/q}}{\omega\eta}$. Hence the mapping is not $1-1$. \square

To study the image of the third curved segment of (2.9) via (2.6) we consider the parametric equations of the curve. These are given by

$$\begin{aligned} x = \operatorname{Re} z &= \frac{1}{\omega\eta^{q/p}} [\cos \theta - (1-\omega)\eta \cos \left(\left(\frac{p-q}{q} \right) \theta \right)] \\ y = \operatorname{Im} z &= -\frac{1}{\omega\eta^{q/p}} [\sin \theta + (1-\omega)\eta \sin \left(\left(\frac{p-q}{q} \right) \theta \right)] \end{aligned} \quad \theta \in [0, \frac{2q\pi}{p}]. \quad (2.10)$$

As is known equations (2.10) define a hypocycloidal curve. Equations (2.10) can be rewritten as follows

$$\begin{aligned} x &= (R-r) \cos t + h \cos \left(\left(\frac{R-r}{r} \right) t \right) \\ y &= (R-r) \sin t - h \sin \left(\left(\frac{R-r}{r} \right) t \right) \end{aligned} \quad t \in [0, 2\pi], \quad (2.11)$$

where R and r are the radii of the large and of the small circle, respectively, and h is the distance from the center of the small circle of the fixed point on the disk of the small circle when this circle rolls in the interior of the large one without sliding. From (2.10) and (2.11) it becomes clear that

$$R-r = \frac{1}{\omega\eta^q}, \quad h = \frac{(\omega-1)\eta^{p-q}}{\omega}, \quad \frac{R-r}{r} = \frac{p-q}{q}. \quad (2.12)$$

A study of the transformation (2.6) using these hypocycloidal curves in a similar way as in [23] follows.

Lemma 2.2. The image of the arc $\{\eta e^{i\theta} : \theta \in [0, \frac{2q\pi}{p}]\}$ via (2.6) is the hypocycloidal curve (2.10). The mapping is 1 – 1 and the curve obtained is symmetric with axis of symmetry the straight line $\{\rho e^{-i\frac{q\pi}{p}} : \rho \in \mathbb{R}\}$.

Proof: Using (2.6) it is easy to prove that the images of two symmetric points wrt the axis $\{\rho e^{i\frac{q\pi}{p}} : \rho \in \mathbb{R}\}$ are symmetric points wrt $\rho e^{i\frac{q\pi}{p}}$. To prove that the mapping is 1 – 1, consider two distinct points $\eta e^{i\theta_1}, \eta e^{i\theta_2}$ with $\theta_1, \theta_2 \in (0, \frac{2q\pi}{p})$. Let the images of these two points coincide. From (2.10), after some operations take place, we obtain that $\sin(\frac{p}{2q}(\theta_1 + \theta_2)) = 0$ or $\theta_1 + \theta_2 = \frac{2kq\pi}{p}$. Since $\theta_1, \theta_2 \in (0, \frac{2q\pi}{p})$, $k = 1$. Also, since θ_1, θ_2 are distinct it is implied that the images are distinct and symmetric wrt the axis $\{\rho e^{iq\pi}, \rho \in \mathbb{R}\}$. This conclusion contradicts our assumption that the images of the two points coincide. Consequently, the mapping is 1 – 1. \square

We distinguish the two cases $\frac{p-q}{q} < 1$ and $\frac{p-q}{q} > 1$. The first inequality implies that the number of nonidentically zero blocks $|\zeta_L|$ of the matrix L is less than that of U , $|\zeta_U|$, while in the second case the situation is reversed. On the other hand, from (2.12), $\frac{p-q}{q} < 1$ implies $R < 2r$ while $\frac{p-q}{q} > 1$ implies $R > 2r$.

After the analysis done so far we give in Figure 1 the various shapes of the corresponding hypocycloidal arcs in all possible cases.

Based on the previous analysis the following theorem can be stated and proved.

Theorem 2.3. The sector $S'_0 := S_0 \setminus \{\rho : \rho \in (0, \eta]\}$ is mapped via (2.6) onto the set of points $R'_0 = z(S'_0)$ which is an open sector of the complex plane described by a semiline by a rotation about the origin through an angle of $-\frac{2q\pi}{p}$ and has as a boundary the image of the curve ∂S_0 via (2.6). Furthermore, the mapping in question is 1 – 1.

Proof: First we define the set $R_0 := z(S_0)$ and note that S'_0 , from which R'_0 was defined, is nothing but S_0 from which the boundary line segments are excluded. This is done because, in view of Lemma 2.1, the mapping for $\eta > \hat{\eta}$ is not 1 – 1. Consider then the point $\zeta_0 = \rho_0 e^{i\theta_0} \in S'_0$. This point can be defined as the intersection of the semiline $\{\rho e^{i\theta_0} : \rho > 0\}$ and of the arc $\{\rho_0 e^{i\theta} : \theta \in (0, \frac{2q\pi}{p})\}$. By virtue of Lemma 2.2 we have that the mapping of the arc in question is a 1 – 1 one. To find out that the mapping of the semiline in question is also 1 – 1 we assume that it is not. This means that there are at least two distinct points of it that have the same image. Let these points be $\zeta_1 = \rho_1 e^{i\theta_0}$ and $\zeta_2 = \rho_2 e^{i\theta_0}$. Equating the expressions for their images, via (2.6), we have that

$$\rho_2 - \rho_1 = (\omega - 1)(\rho_2^{p/q} - \rho_1^{p/q})e^{ip/q\theta_0}. \quad (2.13)$$

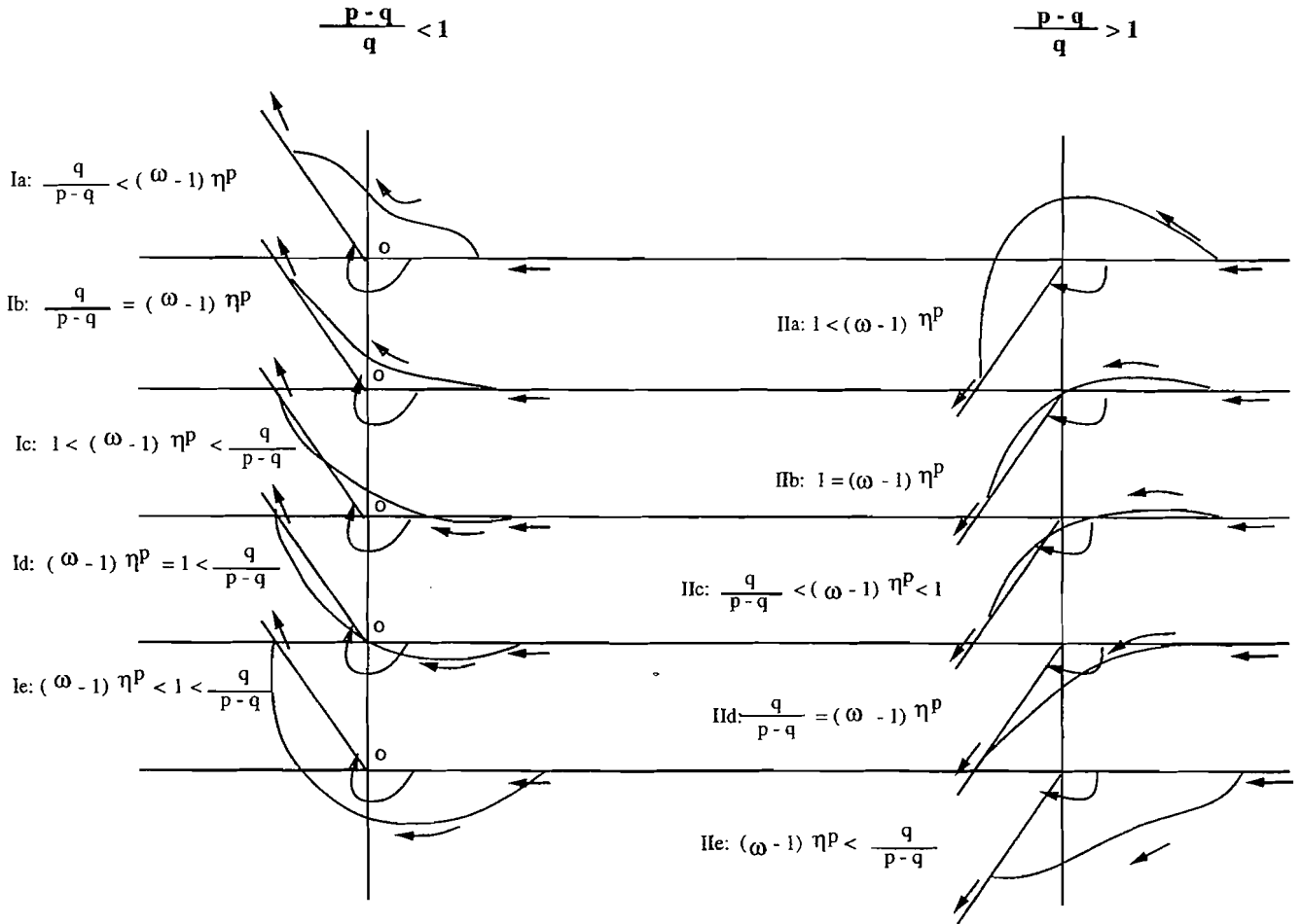


Figure 1:

If $\rho_1 \neq \rho_2$, in (2.13), then we will have equality between a real and a complex number unless $\theta_0 = \frac{\pi q}{p}$ in which case the two members of (2.13) will be real of opposite signs. Hence (2.13) holds iff $\rho_1 = \rho_2$ implying that the mapping is 1 – 1. We have then proved that the image of the point ζ_0 is the unique intersection point z_0 of the images of the two lines considered. Since this holds for any point $\zeta_0 \in S'_0$ the mapping is 1 – 1. Moreover, it is readily checked that R_0 is the left-hand-side part of the complex plane wrt the direction of the arrows, as shown in Figure 1, and the proof is complete. \square

It is also immediately seen, using induction, that the conclusions of Theorem 2.3 hold also for all sectors S'_k , $k = 0(1)p - 1$, in (2.7). This is because for the mappings $R_k := z(S'_k)$, $k = 0(1)p - 1$, there holds $R_k = R_{k-1}e^{-i\frac{2q\pi}{p}}$ $k = 1(1)p - 1$ or

$$R_k = R_0 e^{-i\frac{2kq\pi}{p}}, \quad k = 1(1)p - 1. \quad (2.14)$$

From the analysis so far it is also clear that since the disk $\overline{D}_{\eta^{q/p}}$, $|\zeta| \leq \eta^{q/p}$, is the union of the sectors S_k , $k = 0(1)p - 1$, it will be mapped through (2.6) into $\bigcup_{k=0}^{p-1} R_k$. Consequently, all the points belonging to $\left(\bigcup_{k=0}^{p-1} R_k\right)^c = \bigcap_{k=0}^{p-1} R_k^c$ will be images only of points belonging to the exterior of the disk $\overline{D}_{\eta^{q/p}}$.

Based on the analysis so far the theorem below can be stated and proved.

Theorem 2.4. Let the Jacobi matrix B associated with the linear system (2.1) be block weakly cyclic of index p generated by the cyclic permutation $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_p\}$ with $|\zeta_L| = p - q$. Then for the spectral radius of the corresponding SOR iteration matrix there holds $\rho(\mathcal{L}_\omega) \leq \frac{1}{\eta}$ iff $\sigma(B) \subseteq \bigcap_{k=0}^{p-1} R_k^c$, with R_k being defined in (2.14), $R_0 = z(S_0)$, and z and S_0 in (2.6) and (2.7), respectively. Moreover, $\rho(\mathcal{L}_\omega) = \frac{1}{\eta}$ iff at least one element of $\sigma(B)$ lies on the boundary of $\bigcap_{k=0}^{p-1} R_k^c$.

Proof: If all the eigenvalues of B belong to the region defined above then from the successive transformations (2.1), (2.5), (2.6) and the analysis done so far it follows that all the eigenvalues of \mathcal{L}_ω belong to the closed disk $\overline{D}_{1/\eta}$ with radius $\frac{1}{\eta}$. Conversely, if there exists an eigenvalue $\mu \in \sigma(B)$ such that $\mu \in \bigcup_{k=0}^{p-1} R_k$, then there will exist $i \in \{0, 1, 2, \dots, p - 1\}$ such that $\mu \in R_i$ and from the previous transformations there exists $\lambda \in \sigma(\mathcal{L}_\omega)$ such that $|\lambda| > \frac{1}{\eta}$ and the proof is complete. \square

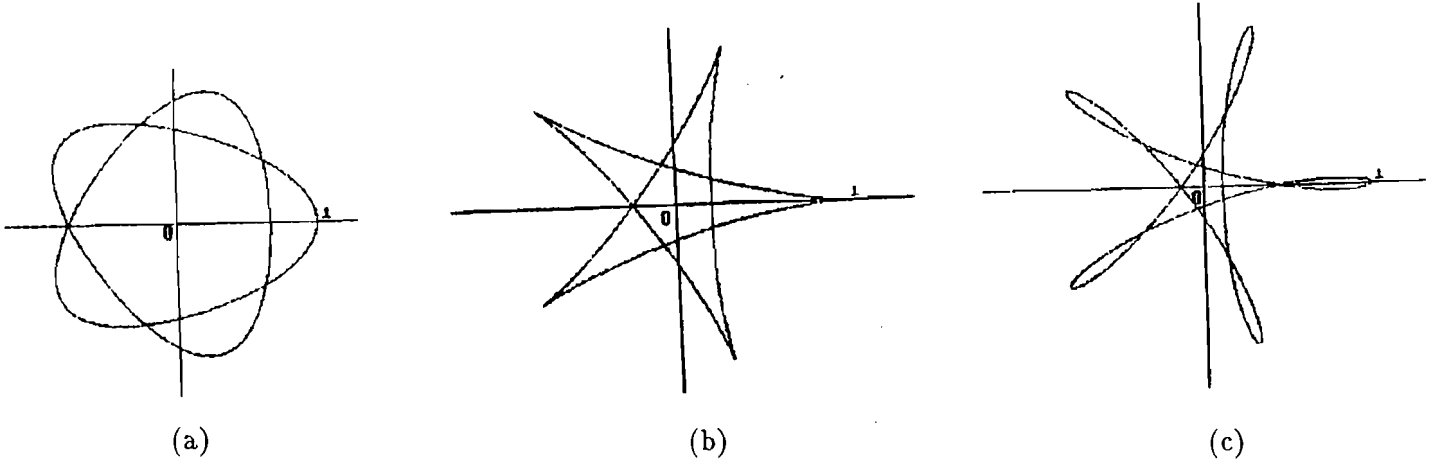


Figure 2:

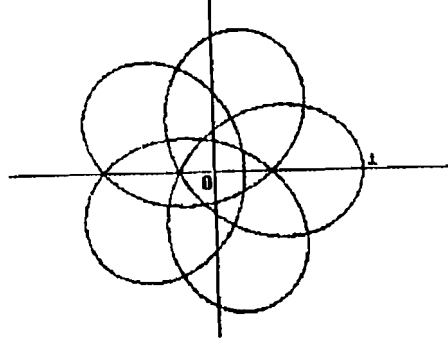


Figure 3:

Remark: We simply remark that in view of Theorem 2.4 and the various cases illustrated in Figure 1, the region of convergence $\bigcap_{k=0}^{p-1} R_k^c$ is the empty set \emptyset in the cases *Ia*, *Ib*, *Ic*, *Id*, *IIa* and *IIb*. So, the cases to be studied in the sequel are the remaining ones of Figure 1 or, equivalently, the ones where

$$\eta < \hat{\eta} = \frac{1}{(\omega - 1)^{\frac{1}{p}}}. \quad (2.15)$$

The regions of convergence for the cases $(p, q) = (5, 2)$ and $(5, 3)$ are illustrated in Figures 2 and 3, respectively.

Before we close this section we mention that in the case where $\omega \in (0, 1)$ we begin with the

transformation (2.6) and follow an analysis analogous to the one in [23], the corresponding regions of convergence can be determined from those obtained for $\omega \in (1, 2)$ by a rotation about the origin through an angle $\frac{q}{p}\pi$. Analogous conclusions to the ones obtained so far can be drawn which are not presented here.

3. Domains of Convergence in the $(\rho(B), \omega)$ -plane

From the previous analysis it has become obvious that the region of convergence remains the same by a rotation about the origin through an angle $\frac{2q}{p}\pi$ or $\frac{2kq}{p}\pi$, $k = 0(1)p - 1$. Since $\text{g.c.d.}(p, q) = 1$, the endpoints of the arcs defining the region of convergence will be p distinct points of a circle separating the circle into p equal consecutive arcs. Hence, rotation of the region of convergence through an angle $\frac{2k\pi}{p}$, $k = 0(1)p - 1$, about the origin will give the same region.

In what follows we study the behavior of the length r of the polar radius from the origin to a point of the boundary of the region of convergence as a function of the polar angle θ . Thus, from the parametric equations (2.10) we have

$$r := |z(\zeta)| = \frac{1}{\omega\eta^{p/q}}[1 + (1 - \omega)^2\eta^2 - 2(1 - \omega)\eta \cos \frac{p}{q}\theta]^{1/2}, \quad \omega \in (1, 2). \quad (3.1)$$

Obviously, r as a function of θ is a decreasing one in $[0, \frac{q}{p}\pi]$ and an increasing one in $[\frac{q}{p}\pi, \frac{2q}{p}\pi]$. So, the smallest value of r , r_{min} , is assumed for $\theta = \frac{q\pi}{p}$ and is given by

$$r_{min} = \frac{1}{\omega\eta^{p/q}}[1 + (1 - \omega)\eta]. \quad (3.2)$$

Because of the cyclic nature of the boundary (curve) of the region of convergence all points of the boundary that have polar angles $\theta = \frac{(2k+1)q}{p}\pi$, $k = 0(1)p - 1$, will have polar radii equal to r_{min} . These arcs correspond to one of the sets of arcs $\{\frac{2k\pi}{p}, k = 0(1)p - 1\}$ or $\{\frac{2k+1}{p}\pi, k = 0(1)p - 1\}$. To the other one will correspond the points of the regions boundary with the maximum absolute value, let it be r_{max} . To find this value we have to find the point of intersection of the boundary curve with the real axis. r_{min} will be the polar radius of the intersection point with the positive semiaxis iff the difference $0 - \frac{q\pi}{p}$ is an integral multiple of $\frac{2\pi}{p}$, or, equivalently, if q is even otherwise the polar radius on the positive semiaxis will be r_{max} . Similarly, r_{min} will correspond to the intersection of the boundary curve with the negative semiaxis if $\pi - \frac{q\pi}{p}$ is an integral multiple of $\frac{2\pi}{p}$ or if $p - q$ is even. Otherwise the intersection with the negative semiaxis will have polar radius r_{max} . We distinguish then three cases:

- i) p even q odd. On both semiaxes the corresponding polar radii will be equal to r_{max} for $\omega > 1$ and equal to r_{min} for $\omega < 1$.
- ii) p odd q even. To the positive semiaxis will correspond a polar radius equal to r_{min} and to the negative one r_{max} .
- iii) p and q odd. To the positive semiaxis will correspond r_{max} and to the negative one r_{min} for $\omega > 1$ while for $\omega < 1$ the situation is reversed.

To conclude the analysis regarding the region of convergence, r_{max} must be determined. For this we begin with (2.10), where in order to find the points of intersection with the real axis we put $y = 0$ to obtain

$$\sin \theta + (1 - \omega)\eta \sin \left(\frac{p-q}{p}\theta \right) = 0, \quad \theta \in [0, 2\pi) \quad (3.3)$$

or, equivalently,

$$U_{q-1}(t) + (1 - \omega)\eta U_{p-q-1}(t) = 0, \quad (3.4)$$

with $t = \cos \frac{\theta}{p}$ and $U_s(t)$ being the Chebyshev polynomial of the second kind of degree s [19]. The intersection points are found from the first equation of (2.10) using the expressions

$$x_i = \frac{1}{\omega\eta^{q/p}}[T_q(t_i) - (1 - \omega)\eta T_{p-q}(t_i)], \quad i = 0(1)p - 2 \quad (3.5)$$

with t_i being the zeros of (3.4) and $T_s(t)$ the Chebyshev polynomial of the first kind of degree s . Then r_{max} will be the smallest value of x_i in cases (i) and (iii) and/or the absolute value of the maximum values of the negative x_i 's in cases (i) and (ii). Apparently, in the general case, this value can only be given numerically.

Here we simply note that in case r_{max} does not belong to the real axis then we rotate the right hand side of (2.6) about the origin through an angle equal to $\frac{q\pi}{p}$ and work in a similar way as in the one we described above.

In the remaining of this section we examine in some detail the two cases of $\sigma(B^p)$ being non-negative and nonpositive.

The first statements that can be proved are the following ones:

Theorem 3.1. Let $\sigma(B^p)$ be nonnegative and the block Jacobi matrix B be generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_p)$ with $|\zeta_L| = p - q$. Then the domain of convergence of the corresponding SOR method in the $(\rho(B), \omega)$ -plane is

$$\Omega_1 = \{(\beta, \omega) : \beta \in [0, 1), \quad 0 < \omega < \omega_1(\beta) = \frac{2}{1 + \beta}\} \quad (3.6)$$

for q even and

$$\Omega_2 = \{(\beta, \omega) : \beta \in [0, 1), \quad 0 < \omega < \omega_2(\beta)\} \quad (3.7)$$

for q odd, where $\omega_2(\beta)$ is a curve in the (β, ω) -plane ($\beta = r_{max}$), is the upper bound of the domain of convergence, and can be given from the relationships (3.4) and (3.5) numerically.

Proof: For q even we necessarily have p odd. So, we are in case (ii) in which we have $\rho(B^p) = r_{min}^p$ or $\rho(B) = r_{min}$. For $\omega > 1$ the value of r_{min} corresponds to an angle $\theta = p \cdot \frac{q}{p} \pi = q\pi$ for $\eta = 1$ and is given from the first equation of (2.10). Hence $\beta = \frac{2-\omega}{\omega}$ or

$$\omega = \frac{2}{1 + \beta} =: \omega_1(\beta). \quad (3.8)$$

For $\omega < 1$, $r_{min} = 1$ and $\theta = 0$, therefore from the first equation of (2.10) we have that

$$\beta = 1. \quad (3.9)$$

From (3.8), (3.9) and the analysis done we obtain that the convergence domain is the one defined in (3.6) and illustrated in Figure 4. For q odd we are in either case (i) or (iii). In both of these cases we have $\rho(B) = r_{max}$ for $\omega > 1$. The value r_{max} can be given from (3.4) and (3.5) as has already been mentioned. For $\omega < 1$, $r_{min} = 1$ and we also have $\beta = 1$. For each $\omega \in (0, 2)$ we obtain a unique value β . Obviously, all the pairs (β, ω) give the boundary curve $\omega_2(\beta)$. \square

We note that the domains Ω_1 and Ω_2 do not coincide and neither of them is a subset of the other.

Theorem 3.2. Under the assumptions of Theorem 3.1, with the only exception being that $\sigma(B^p)$ is nonpositive, the corresponding domains of convergence are

$$\Omega_3 = \{(\beta, \omega) : \omega \in (0, 2), \quad 0 \leq \beta < \beta_1(\omega)\} \quad (3.10)$$

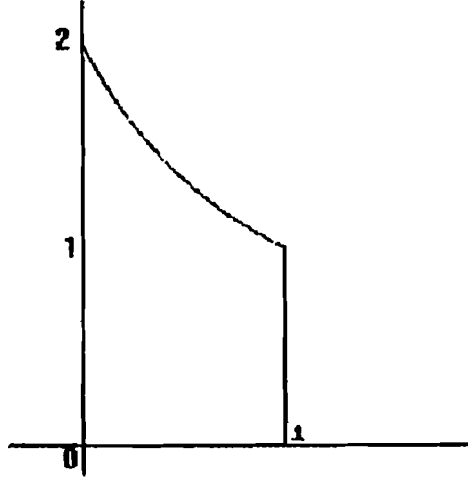


Figure 4:

for q odd. $\beta_1(\omega)$ is a curve in the (β, ω) -plane ($\beta = r_{\max}$) which constitutes the upper bound of Ω_3 and can be found from (3.4) and (3.5) numerically. The curve $\beta_2(\omega)$ is given by $\beta = \frac{2-\omega}{\omega}$ for $\omega > 1$ and can be given from (3.4) and (3.5) for $\omega < 1$. Moreover, $\lim_{\omega \rightarrow 0^+} \beta_1(\omega) = \lim_{\omega \rightarrow 0^+} \beta_2(\omega) = 1/\cos \frac{\pi}{p}$.

Proof: For q even we have $\rho(B^p) = r_{\max}^p$ for both cases $\omega > 1$ and $\omega < 1$, as in Theorem 3.1, and the domain of convergence is Ω_3 given in (3.10). For q odd, we are in cases (i) and (iii). Since $\sigma(B^p)$ is nonpositive, the boundary of this spectrum must be given from the smallest, in absolute value, polar radius of the boundary curve corresponding to the semiaxis with polar angle $\frac{\pi}{p}$. It is clear that this polar radius is r_{\max} if r_{\min} corresponds to the positive semiaxis and vice versa. So, from the cases (i) and (iii) it becomes clear that the polar radius in question is r_{\min} for $\omega > 1$ and r_{\max} for $\omega < 1$. In other words, $\beta_2(\omega) = \frac{2-\omega}{\omega}$ for $\omega > 1$ (or equivalently for $\beta < 1$) and $\beta_2(\omega)$ is found from (3.4) and (3.5) otherwise. It is an immediate conclusion that $\beta_1(2) = \beta_2(2) = 0$ and $\beta_1(1) = \beta_2(1) = 1$. To find the point of intersection of either of the two curves $\beta_1(\omega)$ and $\beta_2(\omega)$ with the axis $\omega = 0$ we have to follow an analysis analogous to the one in the application of stretched hypocycloids in [17]. For $\omega \rightarrow 0^+$ the right boundary of the region of convergence tends to that of a regular polygon as in [17]. Therefore $\beta_{\omega \rightarrow 0^+}(\omega) = \beta_{\omega \rightarrow 0^+}(\omega) \rightarrow 1/\cos \frac{\pi}{p}$ which completes the proof. \square

The domains Ω_3 and Ω_4 for $(p, q) = (5, 2)$ and $(5, 3)$, respectively, are illustrated in Figures 5a and 5b.

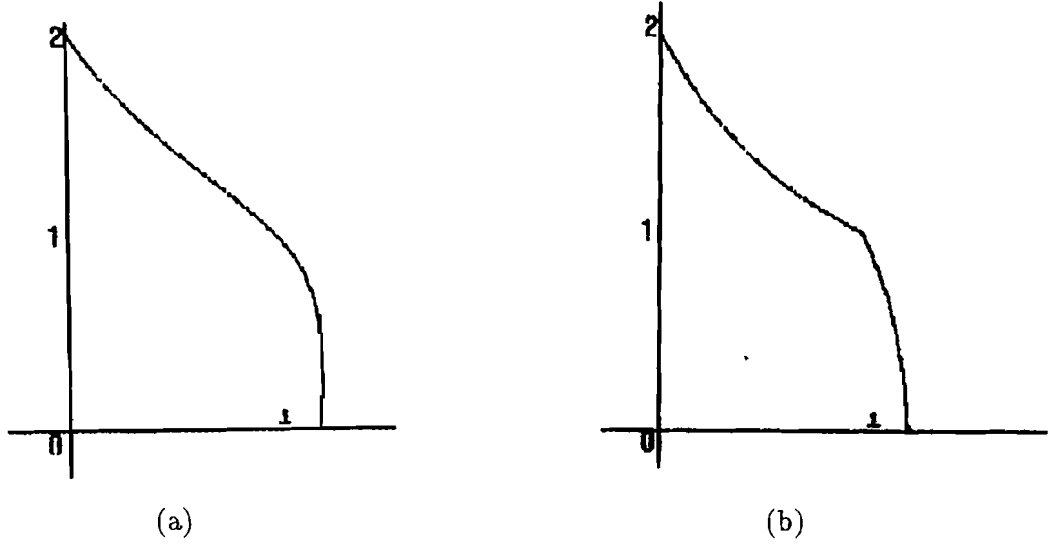


Figure 5:

4. Special Cases

The special cases $(p, q) = (3, 1), (4, 1)$ and $(5, 1)$ as well as $(p, q) = (3, 2), (4, 3)$ and $(5, 4)$ have been studied by many researchers as was mentioned in the introductory Section 1. In this section we study the special cases $(p, q) = (5, 2)$ and $(5, 3)$.

(i) $(p, q) = (5, 2)$: In this case equation (3.5) becomes

$$U_1(t) + (1 - \omega)\eta U_2(t) = 0 \quad (4.1)$$

or

$$4(1 - \omega)\eta t^2 + 2t - (1 - \omega)\eta = 0. \quad (4.2)$$

The roots of (4.2) are

$$t_{+,-} = \frac{-1 \pm \sqrt{1 + 4(1-\omega)^2\eta^2}}{4(1-\omega)\eta} \quad (4.3)$$

In view of (4.3), relationship (3.5) becomes

$$x_{+,-} = \frac{1}{\omega\eta^{5/2}} (T_2(t_{+,-}) - (1-\omega)\eta T_3(t_{+,-})), \quad (4.4)$$

or

$$x_{+,-} = \frac{1}{\omega\eta^{5/2}} (2t_{+,-}^2 - 1 - (1-\omega)\eta(4t_{+,-}^3 - 3t_{+,-})), \quad (4.5)$$

or

$$x_{+,-} = \frac{(1 - (1-\omega)^2\eta^2)(1 \mp \sqrt{1 + 4(1-\omega)^2\eta^2})}{2\omega(1-\omega)^2\eta^{12/5}}. \quad (4.6)$$

The smallest in absolute value of x_+ and x_- gives the value of r_{\max} . It is obvious from (4.6) that $r_{\max} = |x_+|$ and that $x_+ < 0$. The value of r_{\min} is given from (3.2). Therefore

$$r_{\min} = \frac{1 + (1-\omega)\eta}{\omega\eta^{5/2}}. \quad (4.7)$$

Obviously the general theory developed in the previous sections is confirmed in this specific case, where, in addition, we have an analytical expression for r_{\max} .

So, if $\sigma(B^5)$ is nonnegative, by substituting 1 for η we obtain the convergence domain Ω_1 as this is given in Theorem 3.1. On the other hand, if $\sigma(B^5)$ is nonpositive we obtain the convergence domain Ω_3 from Theorem 3.2. In the present case $\beta_1(\omega)$ is given by

$$\beta_1(\omega) = |r_{\max}| = \frac{2(2-\omega)}{1 + \sqrt{1 + 4(1-\omega)^2}}. \quad (4.8)$$

Differentiating $\beta_1(\omega)$ wrt ω it is readily proved that $\beta_1(\omega)$ is a strictly increasing function in $(0, 1/3]$ and a strictly decreasing one in $[1/3, 1) \cup \{1\} \cup (1, 2)$. So,

$$\max_{\omega} \beta_1(\omega) = \beta\left(\frac{1}{3}\right) = \frac{5}{4}.$$

For ω tending to zero we have from (4.8) that

$$\lim_{\omega \rightarrow 0^+} \beta_1(\omega) = \beta_1(0) = \frac{4}{1 + \sqrt{5}} = \frac{1}{\cos \frac{\pi}{5}} \quad (4.9)$$

which confirms the corresponding part of Theorem 3.2.

(ii) $(p, q) = (5, 3)$: Equation (3.5) becomes now

$$U_2(t) + (1 - \omega)\eta U_1(t) = 0 \quad (4.10)$$

or

$$4t^2 + 2(1 - \omega)\eta t - 1 = 0 \quad (4.11)$$

with roots

$$t_{+,-} = \frac{-(1 - \omega)\eta \pm \sqrt{4 + (1 - \omega)^2\eta^2}}{4}. \quad (4.12)$$

As in the previous case, relation (3.5) gives

$$x_{+,-} = \frac{[1 - (1 - \omega)^2\eta^2][(1 - \omega)\eta \mp \sqrt{4 + (1 - \omega)^2\eta^2}]}{2\omega\eta^{3/5}}. \quad (4.13)$$

The value of r_{\max} this time becomes

$$r_{\max} = \begin{cases} |x_+|, & \text{if } \omega \leq 1 \\ |x_-| & \text{if } \omega \geq 1 \end{cases} \quad (4.14)$$

while

$$r_{\min} = \frac{1 + (1 - \omega)\eta}{\omega\eta^{5/3}} \quad (4.15)$$

For the nonnegative case the convergence domain Ω_2 is obtained from Theorem 3.1. The boundary curve $\omega_2(\beta)$ is then given by

$$\beta(\omega) = |r_{\max}| = \frac{1}{2}(2 - \omega)(1 - \omega + \sqrt{4 + (1 - \omega)^2}). \quad (4.16)$$

For the nonpositive case the convergence domain is now Ω_4 given in Theorem 3.2, and $\beta_2(\omega)$ is given by the function

$$\beta_2(\omega) = \begin{cases} \frac{1}{2}(2 - \omega)(\omega - 1 + \sqrt{4 + (1 - \omega)^2}), & \text{if } \omega \geq 1 \\ \frac{2 - \omega}{\omega}, & \text{if } \omega \leq 1. \end{cases} \quad (4.17)$$

It is readily proved that the functions in (4.16), (4.17) are strictly decreasing ones. Also it is obtained that

$$\lim_{\omega \rightarrow 0^+} \beta_2(\omega) = \beta_2(0) = \sqrt{5} - 1 = \frac{1}{\cos \frac{\pi}{5}}, \quad (4.18)$$

confirming the corresponding part of the theory.

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